Solving the Coupled Diffusion Equation

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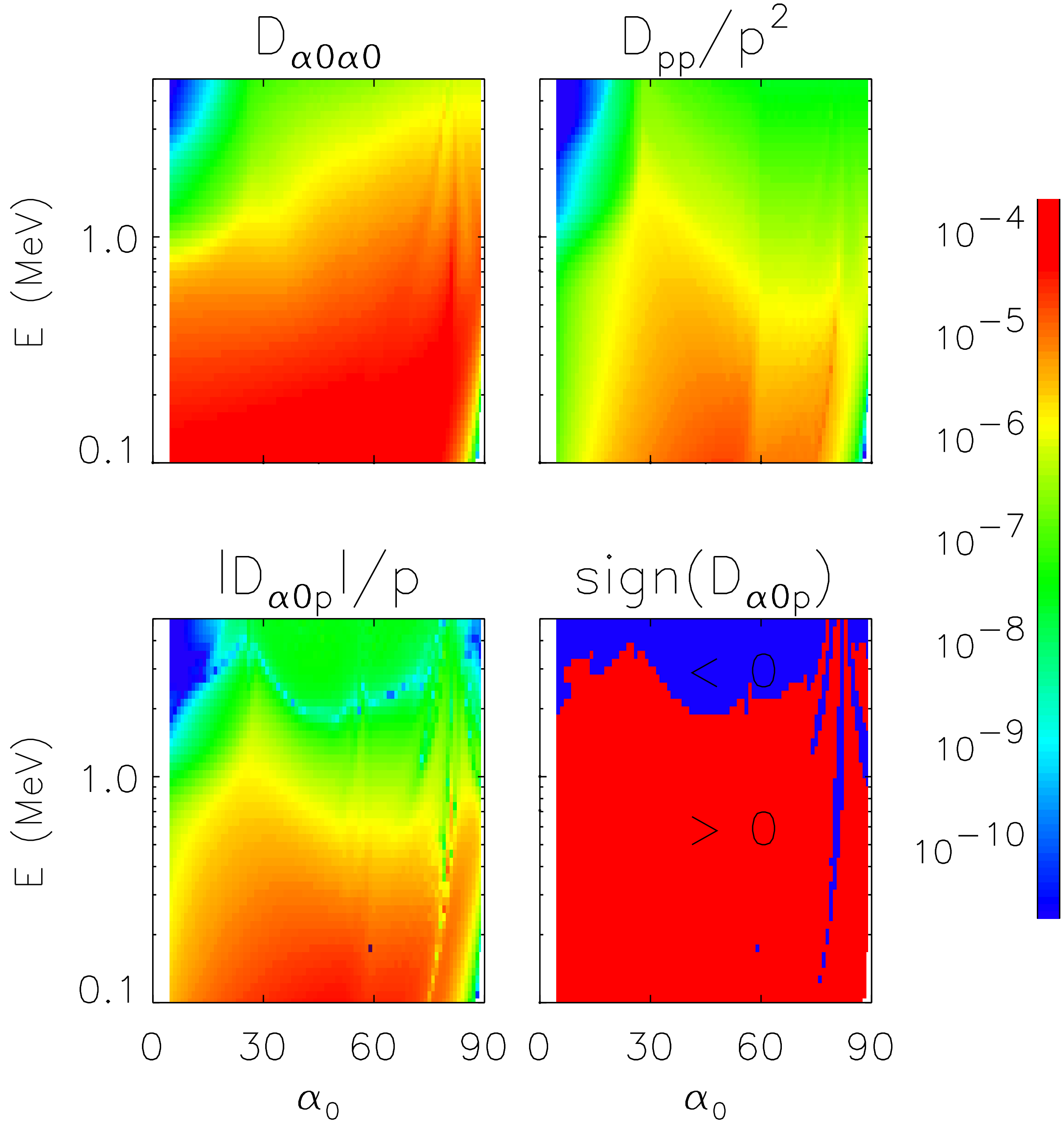
Wave-particle interactions can be very important for electron acceleration and loss. Quasilinear diffusion **coefficients** can be calculated, but we would also like to to solve the diffusion **equation**.

This turns out to be difficult because the cross terms are large and rapidly varying.



D α0α0
D pp/p 2
|D α0p|/p
sign(D α0p)
E (MeV)
1.0
0.1
E (MeV)
1.0
0.1
< 0
> 0
0
30
60
90
0
30
60
90
α 0
α 0
10 −4
10 −5
10 −6
10 −7
10 −8
10 −9
10 −10

Want to solve the diffusion **equation**, $\partial f/\partial t =$

$$\frac{\partial}{\partial J_1}\left[D_{11}\frac{\partial f}{\partial J_1}+D_{12}\frac{\partial f}{\partial J_2}\right]+\frac{\partial}{\partial J_2}\left[D_{12}\frac{\partial f}{\partial J_1}+D_{22}\frac{\partial f}{\partial J_2}\right]+\frac{\partial}{\partial J_3}D_{33}\frac{\partial f}{\partial J_3}$$

or $\partial f/\partial t =$

$$\frac{1}{G}\left\{\frac{\partial}{\partial \alpha_0}G\left[\frac{D_{\alpha_0\alpha_0}}{p^2}\frac{\partial f}{\partial \alpha_0}+\frac{D_{\alpha_0 p}}{p}\frac{\partial f}{\partial p}\right]+\frac{\partial}{\partial p}G\left[\frac{D_{\alpha_0 p}}{p}\frac{\partial f}{\partial \alpha_0}+D_{pp}\frac{\partial f}{\partial p}\right]\right\}$$

$$+L^2\frac{\partial}{\partial L}\frac{D_{LL}}{L^2}\frac{\partial f}{\partial L} \qquad \left(\text{where } G = p^2T(\alpha_0)\sin\alpha_0\cos\alpha_0\right)$$

Treating this with simple finite differencing has **not worked**.

(Local, linear **stability** is not the same as **positivity**.)

Simplest possible finite difference scheme:

$$\frac{f_{ij}^{n+1}-f_{ij}^{n}}{\Delta t}=\left\{D^{11}_{i+\frac{1}{2}j}\frac{f^n_{i+1j}-f^n_{ij}}{\Delta J_1}-D^{11}_{i-\frac{1}{2}j}\frac{f^n_{ij}-f^n_{i-1j}}{\Delta J_1}\right\}\Big/\Delta J_1$$

$$+\left\{D^{12}_{i+1j}\frac{f^n_{i+1j+1}-f^n_{i+1j-1}}{2\Delta J_2}-D^{12}_{i-1j}\frac{f^n_{i-1j+1}-f^n_{i-1j-1}}{2\Delta J_2}\right\}\Big/2\Delta J_1$$

$$+\left\{D^{12}_{ij+1}\frac{f^n_{i+1j+1}-f^n_{i-1j+1}}{2\Delta J_1}-D^{12}_{ij-1}\frac{f^n_{i+1j-1}-f^n_{i-1j-1}}{2\Delta J_1}\right\}\Big/2\Delta J_2$$

$$+\left\{D^{22}_{ij+\frac{1}{2}}\frac{f^n_{ij+1}-f^n_{ij}}{\Delta J_2}-D^{22}_{ij-\frac{1}{2}}\frac{f^n_{ij}-f^n_{ij-1}}{\Delta J_2}\right\}\Big/\Delta J_2,$$

or

$$f_{ij}^{n+1} =$$

$$\frac{D^{11}_{i+\frac{1}{2}j}\Delta t}{\Delta J_1^2} f^n_{i+1j} + \frac{D^{11}_{i-\frac{1}{2}j}\Delta t}{\Delta J_1^2} f^n_{i-1j} + \frac{D^{22}_{ij-\frac{1}{2}}\Delta t}{\Delta J_2^2} f^n_{ij-1} + \frac{D^{22}_{ij+\frac{1}{2}}\Delta t}{\Delta J_2^2} f^n_{ij+1}$$

$$+\left\{1 - \left(\frac{D^{11}_{i+\frac{1}{2}j}\Delta t}{\Delta J_1^2} + \frac{D^{11}_{i-\frac{1}{2}j}\Delta t}{\Delta J_1^2} + \frac{D^{22}_{ij-\frac{1}{2}}\Delta t}{\Delta J_2^2} + \frac{D^{22}_{ij+\frac{1}{2}}\Delta t}{\Delta J_2^2}\right)\right\} f^n_{ij}$$

$$+\frac{(D^{12}_{i+1j} + D^{12}_{ij+1})\Delta t}{4\Delta J_1 \Delta J_2} f^n_{i+1j+1} - \frac{(D^{12}_{i+1j} + D^{12}_{ij-1})\Delta t}{4\Delta J_1 \Delta J_2} f^n_{i+1j-1}$$

$$-\frac{(D^{12}_{i-1j} + D^{12}_{ij+1})\Delta t}{4\Delta J_1 \Delta J_2} f^n_{i-1j+1} + \frac{(D^{12}_{i-1j} + D^{12}_{ij-1})\Delta t}{4\Delta J_1 \Delta J_2} f^n_{j-1j-1}.$$

Because of the cross terms, no $\Delta t > 0$ can guarantee $f_{ij}^{n+1} > 0$.

Solution: change variables, make the cross terms vanish.

Transform from (J_1, J_2) to new variables (Q_1, Q_2):

$$\begin{bmatrix} \tilde{D}_{11} & \tilde{D}_{12} \\ \tilde{D}_{12} & \tilde{D}_{22} \end{bmatrix} = \begin{bmatrix} \frac{\partial Q_1}{\partial J_1} & \frac{\partial Q_1}{\partial J_2} \\ \frac{\partial Q_2}{\partial J_1} & \frac{\partial Q_2}{\partial J_2} \end{bmatrix} \begin{bmatrix} D_{11} & D_{12} \\ D_{12} & D_{22} \end{bmatrix} \begin{bmatrix} \frac{\partial Q_1}{\partial J_1} & \frac{\partial Q_2}{\partial J_1} \\ \frac{\partial Q_1}{\partial J_2} & \frac{\partial Q_2}{\partial J_2} \end{bmatrix}$$

Want (Q_1, Q_2) such that $\tilde{D}_{12} =$

$$\frac{\partial Q_1}{\partial J_1}\frac{\partial Q_2}{\partial J_1} D_{11} + \left(\frac{\partial Q_1}{\partial J_1}\frac{\partial Q_2}{\partial J_2} + \frac{\partial Q_1}{\partial J_2}\frac{\partial Q_2}{\partial J_1} \right) D_{12} + \frac{\partial Q_1}{\partial J_2}\frac{\partial Q_2}{\partial J_2} D_{22} = 0.$$

In general, curves of constant Q_1 are given by

$$dQ_1 = \frac{\partial Q_1}{\partial J_1} dJ_1 + \frac{\partial Q_1}{\partial J_2} dJ_2 = 0, \quad \left.\frac{dJ_2}{dJ_1}\right|_{Q_1=c} = S_1 = -\frac{\partial Q_1/\partial J_1}{\partial Q_1/\partial J_2}$$

and curves of constant Q_2 are given by $dQ_2 = 0$ or

$$\left.\frac{dJ_2}{dJ_1}\right|_{Q_2=c} = S_2 = -\frac{\partial Q_2/\partial J_1}{\partial Q_2/\partial J_2}.$$

The condition $\tilde{D}_{12} = 0$ can be written as

$$\left(\frac{\partial Q_1}{\partial J_2}\frac{\partial Q_2}{\partial J_2}\right)\Big(S_1 S_2 D_{11} - (S_1 + S_2) D_{12} + D_{22}\Big) = 0.$$

Basically, one equation in 2 unknowns, S_1 and S_2.

Free to choose another constraint.

Can choose $S_1 S_2 = -1$, so $\nabla_J Q_1$ is $\perp$ to $\nabla_J Q_2$.

This leads to $\nabla_J Q_1 \| \mathbf{V}$ and $\nabla_J Q_2 \| \mathbf{W}$, the eigenvectors of D_{JJ}:

$$\left.\frac{dJ_2}{dJ_1}\right|_{Q_1=c} = -\frac{V_1}{V_2},$$
$$\left.\frac{dJ_2}{dJ_1}\right|_{Q_2=c} = -\frac{W_1}{W_2}.$$

Simpler to fix S_1 by taking $Q_1 \equiv \alpha_0$.

Then the equation for constant-Q_2 curves,

$$\left.\frac{dJ_2}{dJ_1}\right|_{Q_2=c} = S_2 = \frac{S_1 D_{12} - D_{22}}{S_1 D_{11} - D_{12}}$$

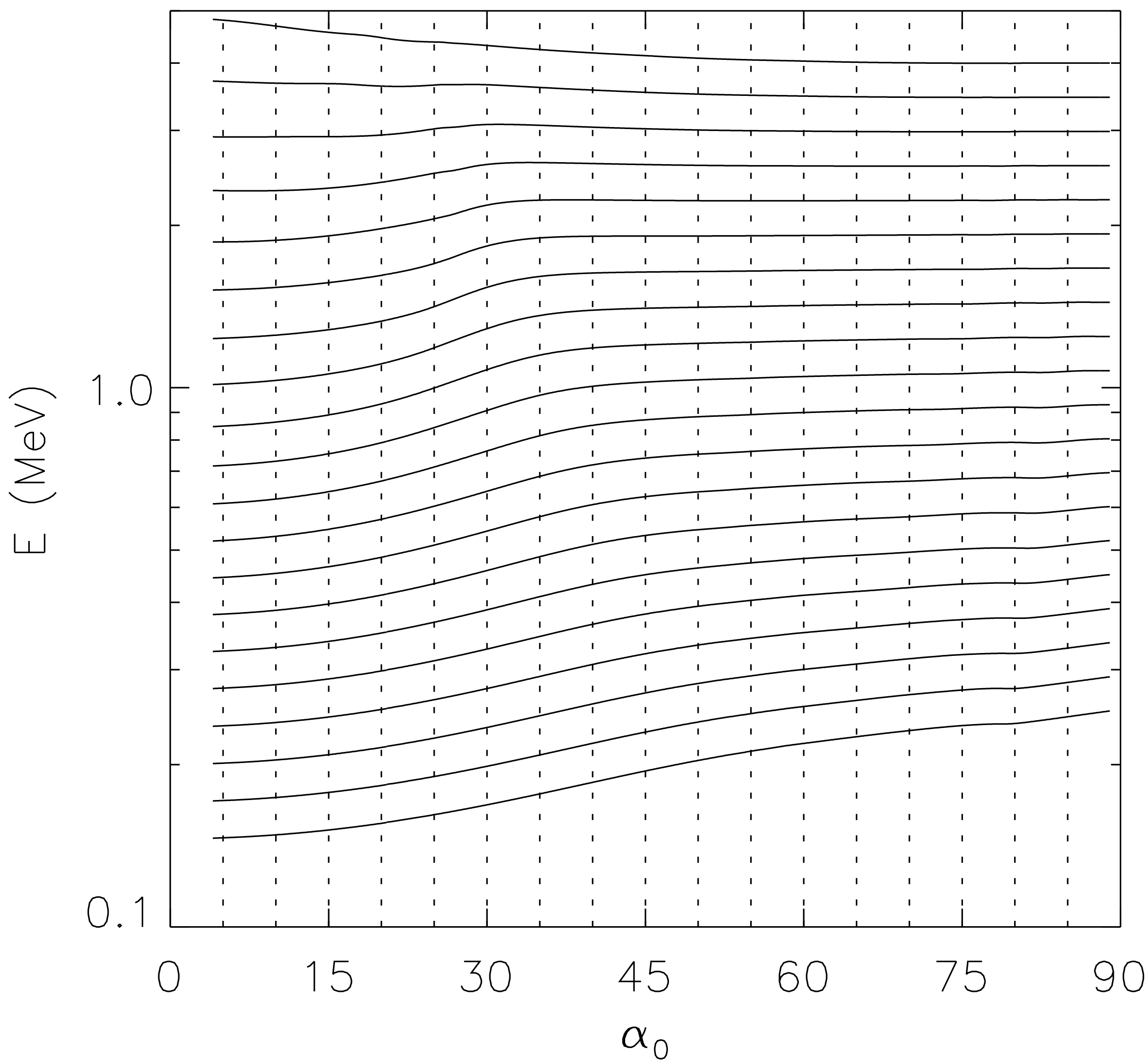
becomes just

$$\frac{dE}{d\alpha_0} = \frac{D_{\alpha_0 E}}{D_{\alpha_0 \alpha_0}}.$$

Integrate this numerically to get curves in the (α_0, E) plane.

The actual values of Q_2 can be any labels of the curves, such as the value of E where the curve goes through $\alpha_0 = 90°$.

To evaluate partials, trace nearby curves and use $\partial E/\partial Q_2 \approx \Delta E/\Delta Q_2$, etc.



E (MeV)
1.0
0.1
0
15
30
45
60
75
90
α_0

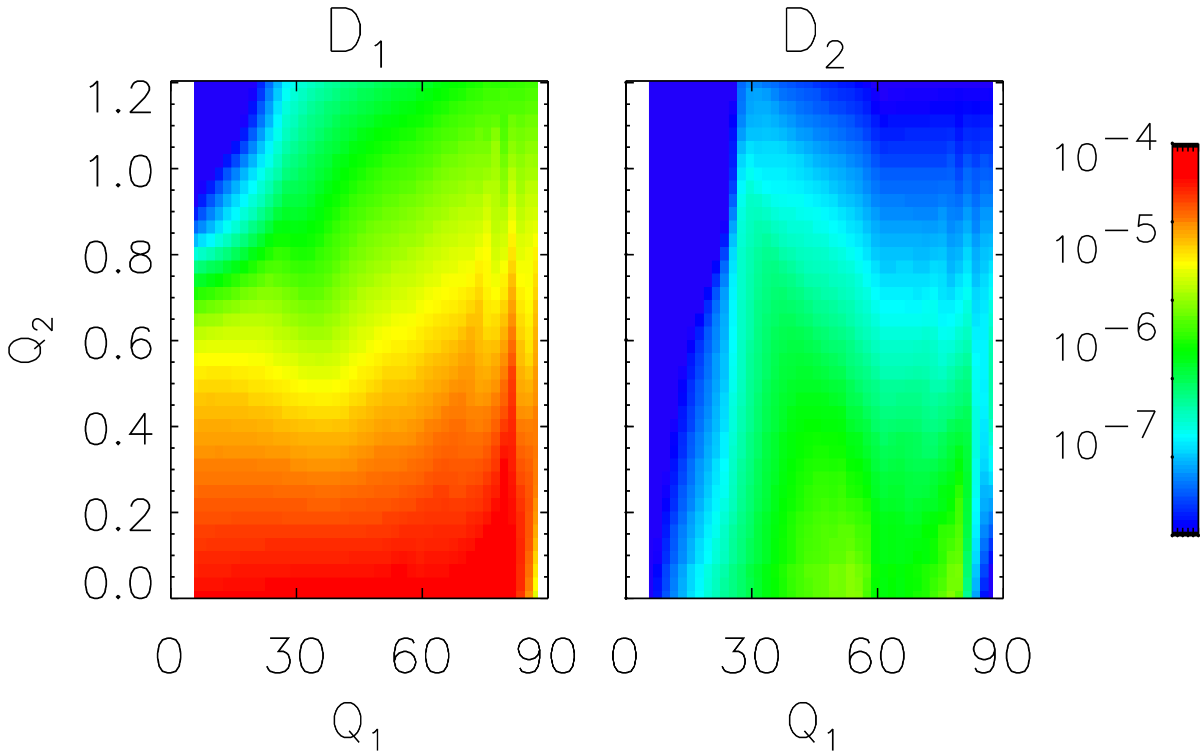
In general, the remaining diffusion coefficients are

$$\tilde{D}_{11} = \left(\frac{\partial Q_1}{\partial J_1}\right)^2 \left(D_{11} - 2D_{12}/S_1 + D_{22}/S_1^2\right),$$
$$\tilde{D}_{22} = \left(\frac{\partial Q_2}{\partial J_2}\right)^2 \left(S_2^2 D_{11} - 2S_2 D_{12} + D_{22}\right).$$

(Both are positive since $D_{11}D_{22} > D_{12}^2$.)

With $Q_1 \equiv \alpha_0$, they become

$$\tilde{D}_{11} = D_{\alpha_0\alpha_0},$$
$$\tilde{D}_{22} = \left(\frac{\partial Q_2}{\partial E}\right)^2 \left(D_{EE} - \frac{D_{\alpha_0 E}^2}{D_{\alpha_0\alpha_0}}\right).$$



D1
D2
1.2
1.0
0.8
0.6
0.4
0.2
0.0
Q2
0
30
60
90
Q1
10−4
10−5
10−6
10−7

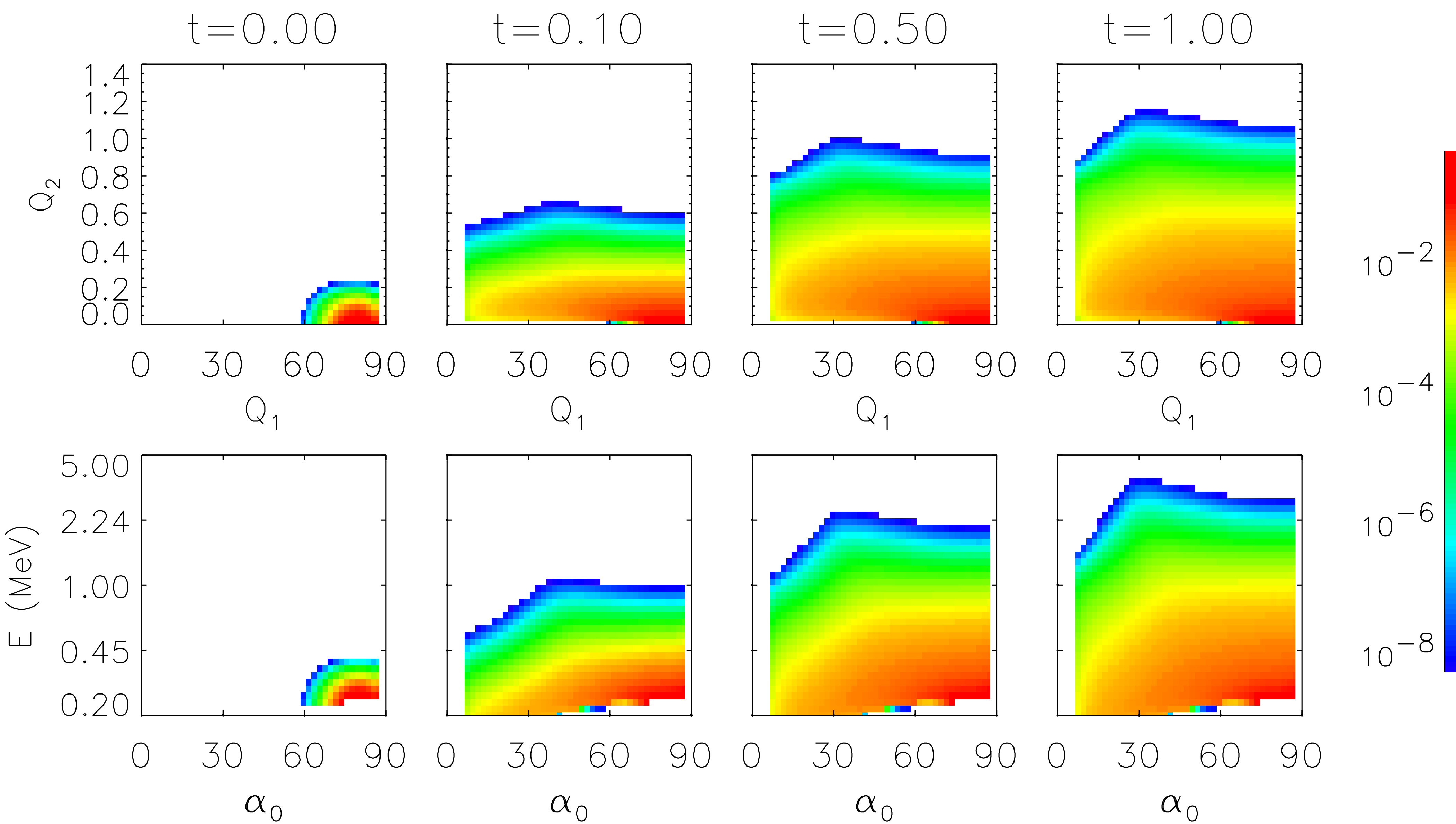
Diffusion in (J_1, J_2) simplifies to

$$\frac{\partial f}{\partial t} = \frac{1}{G}\left(\frac{\partial}{\partial Q_1} G\tilde{D}_{11}\frac{\partial f}{\partial Q_1} + \frac{\partial}{\partial Q_2} G\tilde{D}_{22}\frac{\partial f}{\partial Q_2}\right),$$

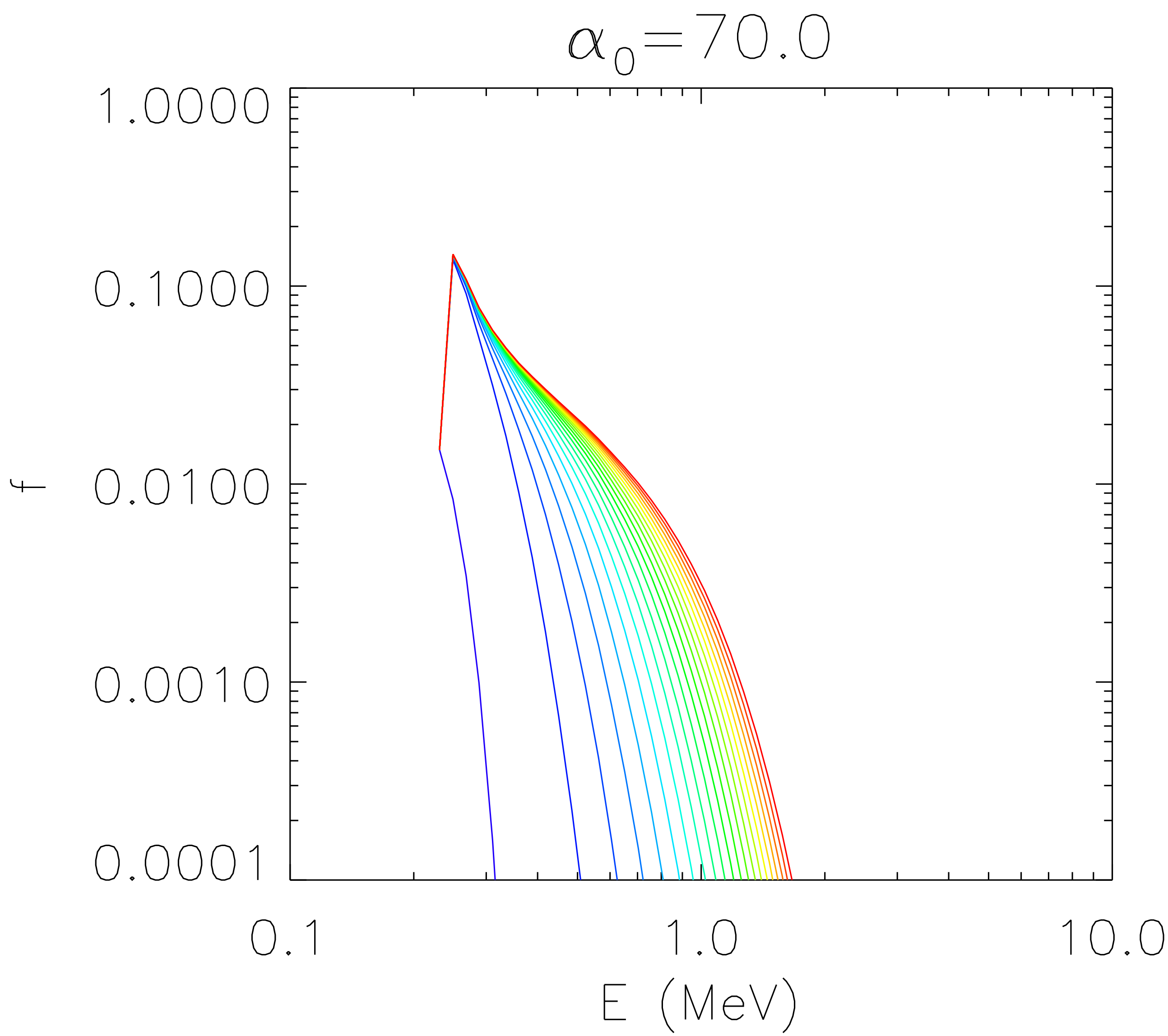
with $G = \det \partial(J_1, J_2)/\partial(Q_1, Q_2)$.

This can be solved easily with finite differencing.

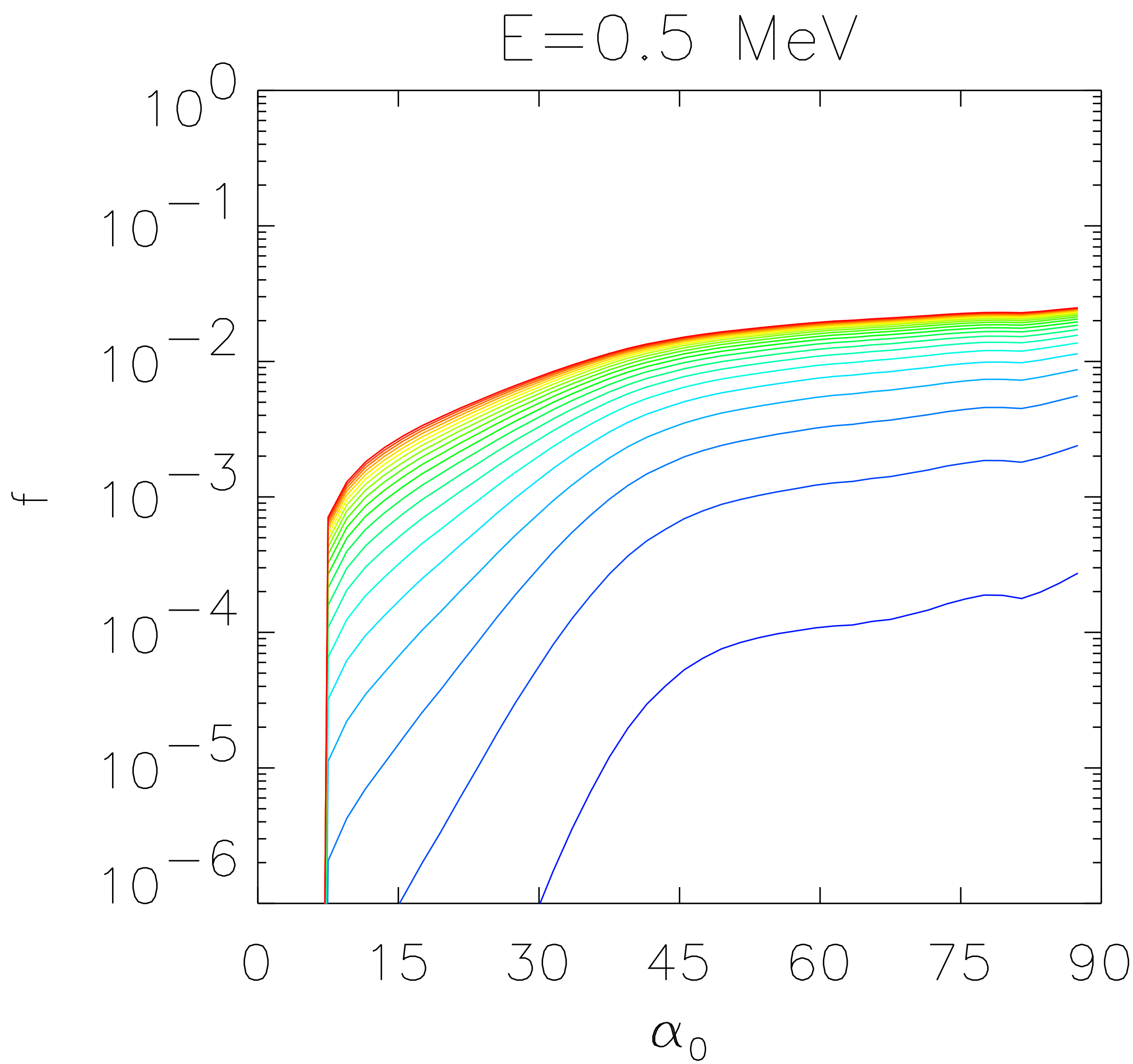
Diffusion in J_3 can be done separately (operator splitting).



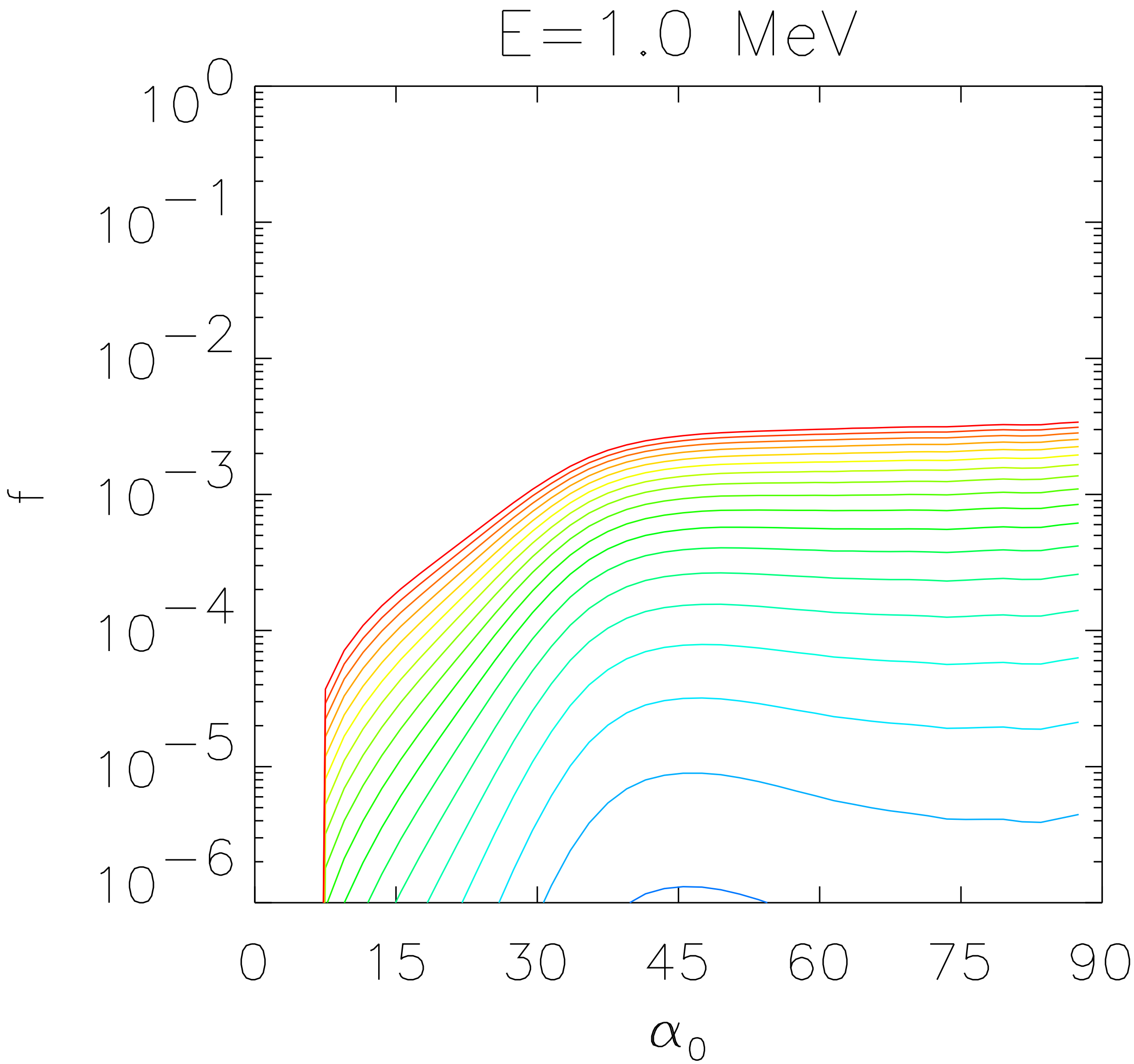
t=0.00
t=0.10
t=0.50
t=1.00
Q2
1.4
1.2
1.0
0.8
0.6
0.4
0.2
0.0
0 30 60 90
Q1
E (MeV)
5.00
2.24
1.00
0.45
0.20
α0
10−2
10−4
10−6
10−8



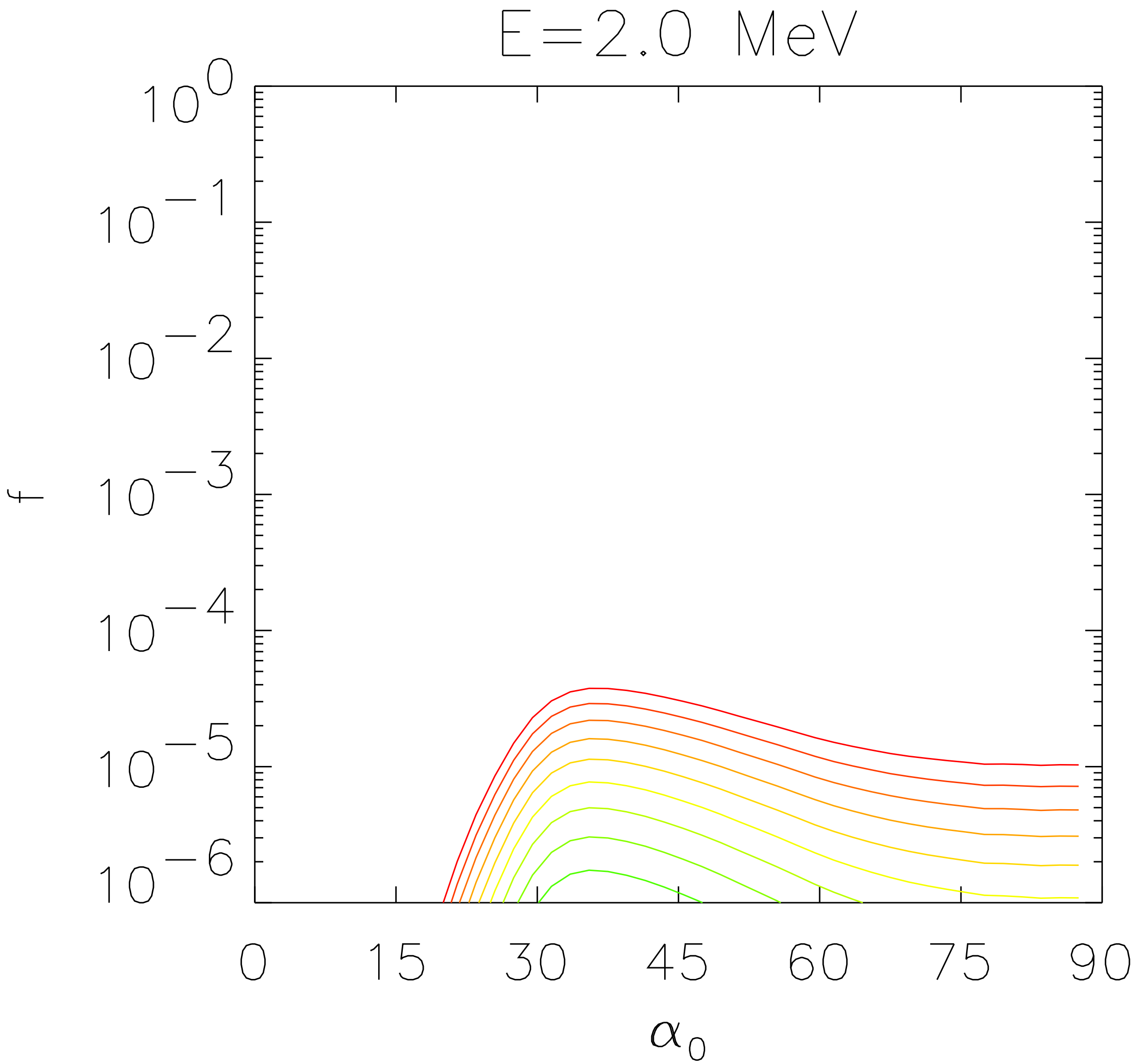
α_0=70.0
1.0000
0.1000
0.0100
0.0010
0.0001
f
0.1
1.0
10.0
E (MeV)



E=0.5 MeV
10^0
10^-1
10^-2
10^-3
10^-4
10^-5
10^-6
f
0
15
30
45
60
75
90
α0



E=1.0 MeV
f
10^0
10^{-1}
10^{-2}
10^{-3}
10^{-4}
10^{-5}
10^{-6}
0
15
30
45
60
75
90
α_0



E=2.0 MeV
f
10^{0}
10^{-1}
10^{-2}
10^{-3}
10^{-4}
10^{-5}
10^{-6}
0
15
30
45
60
75
90
α_0